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A NEW DERIVATION OF THE ADAPTIVE BEAMFORMING FORMULA

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ABSTRACT

The adaptive beamforming formula gives the source power estimate as the reciprocal of an expression similar to the classical frequency domain beamforming formula with the cross spectral matrix (CSM) replaced by an inverse of the CSM. The adaptive formula, also known as Capon and by other names, offers superior resolution, though it can be unstable. The original derivation relies on minimizing the beamformer output power with unity gain in the look direction. Another derivation in the literature is based on maximizing a rank-1 matrix to be subtracted from the cross spectral matrix while keeping the residual non-negative. A third derivation, introduced here, requires only an appropriate statement of the beamforming problem leading to system of linear equations that determines the narrowband source time history. The new derivation may be simpler and more general the previous formulations.

1 INTRODUCTION

Passive beamforming is an array signal processing technique that takes as inputs the signals measured by an array of transducers along with one or more steering vectors. A steering vector defines the amplitude and the propagation delay or, in the frequency domain, the propagation phase between the presumed source and each transducer. The beamforming algorithm attempts to compute the source power and/or the source time history to account for the portion of the signal data that can be attributed to the source corresponding to the steering vector.

The simplest beamforming algorithm in the frequency domain is known as Bartlett beamforming or Frequency Domain Beamforming (FDBF). It suffers from poor resolution at low frequency and high sidelobes.

Several algorithms have been developed to give better results than FDBF. Some operate with one steering vector at time, like FDBF, and others combine the results of beamforming with a grid of steering vectors in a deconvolution procedure to improve the results by a sort of voting. The usual assumption of beamforming is that if there is more than one source contributing to the data, the various sources are mutually incoherent. Inverse methods attempt to treat multiple coherent or partially coherent sources.

Some beamforming methods offer superresolution, meaning they can resolve multiple sources that are separated by less than the central beamwidth of the array using FDBF. The DAMAS and HR-CLEAN-SC deconvolution methods [1,2] have this ability, as do MUSIC [3] and Adaptive Beamforming.

Adaptive Beamforming (AB) is a superresolution method that also known as Capon, Maximum Likelihood, and Minimum Variance among other names [4]. It is usually derived in the literature by applying the method of Lagrange multipliers to solve the constrained optimization problem of finding an array weight vector to minimize the beamformer output while maintaining unity gain in the look direction given by the steering vector. The name reflects that fact that the resulting weight vector depends on the array data.

An alternate derivation AB is to find the largest source strength such that a rank-1 model of the Cross Spectral Matrix (CSM) constructed using a given steering vector can be subtracted from the data CSM giving a result that is still non-negative definite [5]. This derivation can be viewed as an example of "covariance matrix fitting" [6].

Another route to the AB beamforming source power formula, less a derivation than an ansatz, is to set $\nu = -1$ in the Functional Beamforming formula [7].

With an accurate steering vector, the resolution and dynamic range of AB are dramatically better than FDBF. See Fig. 1 for an example plot. It is also lacks the disadvantage of rendering continuous incoherent source distributions of isolated spots as deconvolution methods sometimes do. The computation times for the methods discussed here are not significantly different from FDBF unless the array has hundreds of microphones, in which case the time for the required eigenvalue decomposition of the array CSM can become important.

The high resolution of AB can also cause difficulties. Errors in the computed steering vector can mean that sources are partially or completely missed. Variations on AB that resist this problem are called Robust Adaptive Beamforming. Two approaches include diagonal loading [8,9] and uncertainty sets [5,6].

The intention here is to present a derivation of AD that approaches it from that point of view of fitting a conjugate of a source time history to the narrowband array data. This provides a new way of understanding the method which may broaden and simplify its application. The case of multiple coherent or partially coherent signals is treated using the formulation as an example.

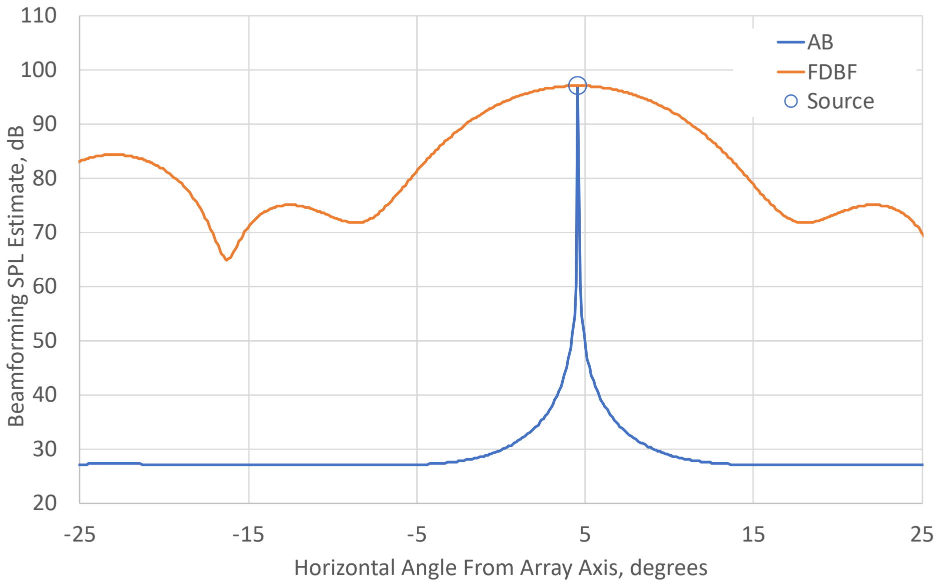

Fig. 1. Sample Point Spread Function using Frequency Domain Beamforming (FDBF) and Adaptive Beamforming (AB). 40 element 2D array, 30 cm aperture, 8 kHz.

1.1 Problem and FDBF

Consider a phased array with N microphones or other transducers that has been used to measure T frequency-domain snapshots of the narrowband acoustic pressure, giving a complex $N \times T$ data matrix, $\boldsymbol{P}$. The idealized picture of the beamforming problem is that there are M acoustic sources, all of which are mutually incoherent. Each source has an $N \times 1$ array steering vector, $\boldsymbol{g}_j, j = 1, \ldots, M$. These steering vectors are assumed to be normalized to unity, and are not required to be mutually orthogonal. Over the T snapshots in time, each source has a $T \times 1$ conjugate time history, $\boldsymbol{q}_j, j = 1, \ldots, M$. The source-receiver model is

$$\boldsymbol{P} = \sum_{j=1}^{M} \boldsymbol{g}_j \boldsymbol{q}_j' \tag{1}$$

where $'$ is the Hermitian conjugate. This model can include spherical wave sources, directive sources, and even microphone self-noise by using a $\boldsymbol{g}_j$ that is non-zero at only one microphone. The elements of the row vectors $\boldsymbol{q}_j'$ are understood as samples of stationary, statistical, random processes representing the sources. The column vectors $\boldsymbol{q}_j$, the conjugates of the $\boldsymbol{q}_j'$, are important in the analysis. The steering vectors $\boldsymbol{g}_j$ are constant and are determined by the source locations and other fixed characteristics, as well as source-receiver propagation paths and the transducers.

The average powers of sources are defined by

$$s_j = \frac{1}{T} \boldsymbol{q}_j' \boldsymbol{q}_j, \quad j = 1, \ldots, M \tag{2}$$

The mutual incoherence of the sources assumed in this case means that $\boldsymbol{q}'_k\boldsymbol{q}_j = 0$ if $k \neq j$.

The sample estimate of the cross spectral matrix (CSM) $\boldsymbol{C}$ is given by

$$\boldsymbol{C} = \frac{1}{T}\boldsymbol{P}\boldsymbol{P}'. \tag{3}$$

Combing Eqs. (1-3) gives

$$\boldsymbol{C} = \sum_{j=1}^{M} s_j \boldsymbol{g}_j \boldsymbol{g}'_j. \tag{4}$$

The beamforming problem for the source power is to use a measured CSM, $\boldsymbol{C}$, and a given steering vector, $\boldsymbol{g}$, which may or may not be one of the true steering vectors, $\boldsymbol{g}_j$, to produce an estimate of the corresponding source strength, b. If $\boldsymbol{g} = \boldsymbol{g}_j$, then is it hoped that $b \approx s_j$. A straightforward least-squares fit of $b\boldsymbol{g}\boldsymbol{g}'$ to Eq. (4) gives the FDBF estimate

$$b = \boldsymbol{g}'\boldsymbol{C}\boldsymbol{g}. \tag{5}$$

A secondary problem is to estimate the source time history, $\boldsymbol{q}$ to go with $\boldsymbol{g}$ by processing $\boldsymbol{P}$. Performing a snapshot by snapshot least-squares fit to estimate each of the T elements of $\boldsymbol{q}$ from the corresponding time slices of $\boldsymbol{P}$ gives

$$\boldsymbol{q}' \approx \boldsymbol{g}'\boldsymbol{P}. \tag{6}$$

Incidentally, substituting Eq. (6) into Eq. (2) gives an alternate derivation of Eq. (5).

2 IMPROVED FORMULA

2.1 Specific problem statement

Suppose the narrowband time history $\boldsymbol{P}$ and a specific steering vector, $\boldsymbol{g}$, are given and the goal is to determine the corresponding conjugate time series, $\boldsymbol{q}$. The expression $\boldsymbol{g}\boldsymbol{q}'$ is intended as a model to explain part of $\boldsymbol{P}$. In particular, the goal is to find a $\boldsymbol{q}$ to approximately solve

$$\boldsymbol{P} = \boldsymbol{g}\boldsymbol{q}' + \boldsymbol{E}, \tag{7}$$

where $||\boldsymbol{q}||$ is maximized and $||\boldsymbol{E}\boldsymbol{q}||$ is small. Maximizing $||\boldsymbol{q}||$ maximizes the amount of $\boldsymbol{P}$ that $\boldsymbol{g}\boldsymbol{q}'$ explains. Minimizing $||\boldsymbol{E}\boldsymbol{q}||$ means that the time history of the residual, $\boldsymbol{P} - \boldsymbol{g}\boldsymbol{q}'$, is approximately incoherent with $\boldsymbol{q}$. This is consistent with the idea that residual is associated with sources that are different from, incoherent with, the source that is described by $\boldsymbol{g}$ and $\boldsymbol{q}$.

To demonstrate the feasibility of second part of the goal, suppose that Eq. (1) holds and that $\boldsymbol{g}$ happens to be an exact steering vector $\boldsymbol{g}_k$ so that

$$\boldsymbol{P} = \boldsymbol{g}_k\boldsymbol{q}' + \sum_{j\neq k} \boldsymbol{g}_j\boldsymbol{q}_j'. \tag{8}$$

Choosing $\boldsymbol{q} = \boldsymbol{q}_k$ gives $\boldsymbol{E} = \sum_{j\neq k} \boldsymbol{g}_j\boldsymbol{q}_j'$. Then the source time history orthogonality gives $\|\boldsymbol{Eq}\| = 0$. This result does not require orthogonality of any steering vectors, only that Eq. (1) holds and $\boldsymbol{g}$ is well constructed so that $\boldsymbol{g} \in \{\boldsymbol{g}_j\}$.

2.2 Finding the conjugate source time history

Multiplying Eq. (7) on the right by $\boldsymbol{q}$ gives

$$\boldsymbol{Pq} = \boldsymbol{g}sT + \boldsymbol{Eq}, \tag{9}$$

where the average source strength s is defined by $\boldsymbol{q}'\boldsymbol{q} = sT$. Dividing both sides by sT gives

$$\boldsymbol{Px} = \boldsymbol{g} + \frac{\boldsymbol{Eq}}{\|\boldsymbol{q}\|^2}, \tag{10}$$

where

$$\boldsymbol{x} = \frac{\boldsymbol{q}}{sT}. \tag{11}$$

Since the problem is to maximize $\|\boldsymbol{q}\|$ and minimize $\|\boldsymbol{Eq}\|$, a good place to start is to find a least-squares solution, $\overline{\boldsymbol{x}}$, to $\boldsymbol{Px} = \boldsymbol{g}$. This minimizes the square error $\|\boldsymbol{Px} - \boldsymbol{g}\|^2 = \frac{\|\boldsymbol{Eq}\|^2}{\|\boldsymbol{q}\|^4}$.

If $T > N$, then the $N \times T$ system of linear equations $\boldsymbol{Px} = \boldsymbol{g}$ is probably underdermined, so it has more than one least-squares solution. In this case, the one to choose is the least-squares solution of minimum norm. Minimizing the norm of $\boldsymbol{x}$ over the subspace of least squares solutions has the desired result of maximizing $\|\boldsymbol{q}\|$, since $\|\boldsymbol{q}\| = \frac{1}{T\|\boldsymbol{x}\|}$.

Let the Moore-Penrose inverse of $\boldsymbol{P}$ be denoted $\boldsymbol{P}^+$. Using the SVD, it can be evaluated as $\boldsymbol{P}^+ = \boldsymbol{V\Sigma}^+\boldsymbol{U}'$. The least-squares solution of minimum norm of $\boldsymbol{Px} = \boldsymbol{g}$ is $\overline{\boldsymbol{x}} = \boldsymbol{P}^+\boldsymbol{g}$. This gives the beamforming conjugate source time history to accompany $\boldsymbol{g}$ as

$$\boldsymbol{q} = sT\,\overline{\boldsymbol{x}} = sT\boldsymbol{P}^+\boldsymbol{g}. \tag{12}$$

2.3 Evaluating the source power

Equation (12) does not fully define the source time history because s appears explicitly on the RHS and implicitly on the LHS. To resolve this, note that

$$\boldsymbol{q}' = Ts\boldsymbol{g}'\boldsymbol{P}^{+\prime}. \tag{13}$$

Combining,

$$s = \frac{1}{T}\boldsymbol{q}'\boldsymbol{q} = s^2T\boldsymbol{g}'\boldsymbol{P}^{+\prime}\boldsymbol{P}^+\boldsymbol{g} \tag{14}$$

Using the SVD of $\boldsymbol{P}$ in Eq. (3) gives

$$\boldsymbol{C} = \frac{1}{T}\boldsymbol{U\Sigma\Sigma' U'}. \tag{15}$$

The Moore-Penrose inverse of $\mathbf{C}$ is therefore

$$\boldsymbol{C}^{+} = T\boldsymbol{U}(\boldsymbol{\Sigma\Sigma'})^{+}\boldsymbol{U}' = T\boldsymbol{P}^{+\prime}\boldsymbol{P}^{+}. \tag{16}$$

Using this in Eq. (13) gives

$$s = s^2\boldsymbol{g}'\,\boldsymbol{C}^{+}\boldsymbol{g}. \tag{17}$$

Solving for s:

$$s = \frac{1}{\boldsymbol{g}'\,\boldsymbol{C}^{+}\boldsymbol{g}}\,. \tag{18}$$

This is recognized as the adaptive beamforming formula [1]. This also completes the determination of $\boldsymbol{q}$.

2.4 Revisiting the time history

Combining Eqs. (13) and (18) gives

$$\boldsymbol{q}' = \frac{T}{\boldsymbol{g}'\,\boldsymbol{C}^{+}\boldsymbol{g}}\boldsymbol{g}'\boldsymbol{P}^{+\prime}\,. \tag{19}$$

Equation (16) and two properties of the Moore-Penrose inverse, $\boldsymbol{P}^{+}\boldsymbol{P}\boldsymbol{P}^{+} = \boldsymbol{P}^{+}$ and $(\boldsymbol{P}^{+}\boldsymbol{P})' = \boldsymbol{P}^{+}\boldsymbol{P}$, can be used to derive the useful formula

$$\boldsymbol{P}^{+\prime} = \tfrac{1}{T}\boldsymbol{C}^{+}\boldsymbol{P}. \tag{20}$$

Substituting this into Eq. (19) gives

$$\boldsymbol{q}' = \frac{\boldsymbol{g}'\boldsymbol{C}^{+}}{\boldsymbol{g}'\,\boldsymbol{C}^{+}\boldsymbol{g}}\boldsymbol{P} = \boldsymbol{h}'\boldsymbol{P}, \tag{21}$$

where

$$\boldsymbol{h} = \frac{\boldsymbol{C}^{+}\boldsymbol{g}}{\boldsymbol{g}'\,\boldsymbol{C}^{+}\boldsymbol{g}}\,. \tag{22}$$

This is analogous to Eq. 6, except that $\boldsymbol{g}$ is replaced by $\boldsymbol{h}$, the familiar weight vector in adaptive beamforming. In this form, it is seen that the method can be applied to determine $\boldsymbol{q}$ and s without actually taking the SVD of $\boldsymbol{P}$. The only decomposition necessary is the spectral decomposition of $\mathbf{C}$, which is used to compute $\mathbf{C}^{+}$.

In retrospect, Eq. (6) (FDBF) can be derived from Eq. (7) by multiplying both sides on the left by $\boldsymbol{g}'$ and assuming that the columns of $\boldsymbol{E}$ are perpendicular to $\boldsymbol{g}$. The adaptive method is derived by multiplying both sides on the right by (as yet unknown) $\boldsymbol{q}$ and assuming that the rows of $\boldsymbol{E}$ are perpendicular to $\boldsymbol{q}$. The difference is that the adaptive assumption has a good chance of being true, whereas the FDBF assumption is unlikely. The two outcomes are compared in in the power sense Fig. 1.

3 MATRIX GENERALIZATION

3.1 Setup

The standard beamforming approach is to determine the steering vectors for the mutually incoherent sources a priori and fit the data to the steering vectors. In some cases, a source is expected to excite several steering vectors with relative contributions and partial coherence that cannot be predicted in advance. Suppose, for example, that an aeroacoustic source at a given location can radiate in three dipole modes. The dipoles should be coherent because they originate from a single source location that is associated with a single, random, pattern of turbulent flow. There are other sources at other locations that are incoherent with the source of interest. Equation (7) is then generalized to

$$\boldsymbol{P} = \boldsymbol{g}_1\boldsymbol{q}_1' + \boldsymbol{g}_2\boldsymbol{q}_2' + \boldsymbol{g}_3\boldsymbol{q}_3' + \boldsymbol{E} \tag{23}$$

where $\boldsymbol{g}_1$ is the known array steering vector for a dipole at the location or interest, oriented in the x-direction, and so on for the y- and z-directions. The time history functions $\boldsymbol{q}_1'$, $\boldsymbol{q}_2'$, and $\boldsymbol{q}_3'$ are partially or fully coherent with each other, but incoherent with the other sources represented by $\boldsymbol{E}$.

Monopoles and quadrupoles may be important, in addition to dipoles. In another application, different parts of a vibrating structure may radiate waves with known patterns, and the parts are different acoustic sources, but are correlated because they are all driven by the structure. Multipath can give rise to coherent sources. In a different field, electromagnetic antenna sources can have two correlated polarizations. Generalizing the number of partially coherent sources from 3 to L and placing $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_L$ on the columns of an $N \times L$ matrix, $\boldsymbol{G}$, and $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_L$ on the columns of a $T \times L$ matrix, $\boldsymbol{Q}$, the expanded version of Eq. (7) becomes

$$\boldsymbol{P} = \boldsymbol{G}\boldsymbol{Q}' + \boldsymbol{E} \tag{24}$$

where, again, $\boldsymbol{P}$ is the $N \times T$ matrix of array data snapshots and the intention is to find sources $\boldsymbol{Q}$ that are consistent with $\boldsymbol{G}$ and incoherent with the sources in $\boldsymbol{E}$.

3.2 Matrix solution

Multiplying Eq. (24) on the right by $\boldsymbol{Q}$ gives

$$\boldsymbol{P}\boldsymbol{Q} = \boldsymbol{G}\boldsymbol{S}T + \boldsymbol{E}\boldsymbol{Q} \tag{25}$$

where $\boldsymbol{S}$ is the $L \times L$ cross spectral matrix of the L sources,

$$\boldsymbol{S} = \frac{1}{T}\boldsymbol{Q}'\boldsymbol{Q} \tag{26}$$

Multiplying Eq. (25) on the right by $\boldsymbol{S}^{-1}$ and introducing

$$\boldsymbol{X} = \frac{1}{T}\boldsymbol{Q}\boldsymbol{S}^{-1} \tag{27}$$

causes Eq. (25) to become

$$\boldsymbol{P}\boldsymbol{X} = \boldsymbol{G} + \boldsymbol{E}\boldsymbol{Q}\,(\boldsymbol{Q}'\boldsymbol{Q})^{-1}. \tag{28}$$

Following the pattern for the single-$\boldsymbol{g}$ case above, the least-squares solution of minimum norm to $\boldsymbol{P}\boldsymbol{X} = \boldsymbol{G}$ is computed as

$$\overline{\boldsymbol{X}} = \boldsymbol{P}^{+}\boldsymbol{G} \tag{29}$$

and

$$\boldsymbol{Q} = T\overline{\boldsymbol{X}}\boldsymbol{S} \quad = \quad T\boldsymbol{P}^{+}\boldsymbol{G}\boldsymbol{S}. \tag{30}$$

Combining Eqs. (16), (26) and (30) and using the fact that $\boldsymbol{S}$ is Hermitian gives $\boldsymbol{S} = \boldsymbol{S}\boldsymbol{G}'\mathbf{C}^{+}\boldsymbol{G}\boldsymbol{S}$. The generalization of Eq. (18) then becomes

$$\boldsymbol{S} = (\boldsymbol{G}'\mathbf{C}^{+}\boldsymbol{G})^{-1}. \tag{31}$$

This is analogous to the FSBF version $\boldsymbol{S}_{\text{FDBF}} = \boldsymbol{G}'C\boldsymbol{G}$, but expected to have much higher resolution.

To finalize the matrix source time history, Eq. (31) is substituted into Eq. (30) and Eq. (20) is used again, giving

$$\boldsymbol{Q}' = \boldsymbol{H}'\boldsymbol{P}, \tag{32}$$

where

$$\boldsymbol{H} = \mathbf{C}^{+}\boldsymbol{G}(\boldsymbol{G}'\mathbf{C}^{+}\boldsymbol{G})^{-1} \tag{33}$$

is the matrix generalization of the adaptive beamforming weight vector.

4 IMPLEMENTATION DETAILS

At several points, the methods depends on computing $\boldsymbol{C}^{+}\boldsymbol{g}$. In the notation used for the SVD of $\boldsymbol{P}$,

$$\boldsymbol{C} = T\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{U}' = T\boldsymbol{U}\mathrm{diag}(\lambda_1, \dots, \lambda_N)\boldsymbol{U}', \tag{34}$$

where the $\lambda_i = \sigma_i^2, i = 1, \dots, N$. Hence

$$\boldsymbol{C}^+ = \frac{1}{T}\boldsymbol{U}[\mathrm{diag}(\lambda_1, \dots, \lambda_N)]^+\boldsymbol{U}'. \tag{35}$$

In the pseudoinverse, $[\mathrm{diag}(\lambda_1, \dots, \lambda_N)]^+$ is formed by taking the reciprocal of the nonzero λ_i values and leaving the 0 values unchanged. For adaptive beamforming, it is not good to leave any zeros on the diagonal of the result, since that would invite zero results in the denominator of, for example, Eq. (18). One solution is to add a small, positive, diagonal matrix to $\boldsymbol{C}$ before taking the spectral decomposition. Since $\boldsymbol{C}$ is non-negative definite by construction, adding the positive matrix ensures that none of the eigenvalues of the sum are 0. One disadvantage of this approach is that it changes all of the eigenvalues, including those that were accurate before they were changed.

The pseudoinverse $\boldsymbol{C}^+$often has some very large eigenvalues, corresponding to the noise subspace in $\mathbb{C}^N$. If $\boldsymbol{g}$ is in this subspace, then $\boldsymbol{g}'\,\boldsymbol{C}^+\boldsymbol{g}$ will be very large and the adaptive beamforming results will be very small. This gives the method its high resolution and dynamic range, but can also create serious problems if the none of the assumed steering vectors are in the signal subspace. The standard solution is diagonal loading: the diagonal matrix added to $\boldsymbol{C}$ before taking the generalized inverse is increased until the method begins to produce results [9]. Another approach is to optimize $\boldsymbol{g}$ within a small region to maximize the output [5,6]

5 SUMMARY

A new derivation of Adaptive Beamforming is based on linear algebra to find the source time history, instead of constrained optimization to find the source power, s. The method is to model the time history as the sum of the source of interest, which depends on the known steering vector(s) of interest and the remaining sources, $\boldsymbol{E}$, which are assumed to be incoherent with the source of interest.

The model equation is used to derive a system of linear equations for a vector $\boldsymbol{x}$. The matrix in the liner system is the array data matrix, $\boldsymbol{P}$ and the right hand side is either the steering vector for the source of interest, $\boldsymbol{g}$, or a matrix of these steering vectors. The least-squares solution of minimum norm is found using the Moore-Penrose inverse of $\boldsymbol{P}$. The new variable, $\boldsymbol{x}$, is parallel to the conjugate source time history, but the norm of $\boldsymbol{x}$ is the reciprocal of the norm of the source time history. Finding the least squares solution $\boldsymbol{x}$ for makes the final source time history approximately orthogonal to $\boldsymbol{E}$. Minimizing the norm of $\boldsymbol{x}$ maximizes to the norm of the source time history, which maximizes the extent to which the model accounts for $\boldsymbol{P}$. The solution, $\overline{\boldsymbol{x}}$, is used to find the source power and the source time history.

The derivation does not explicitly rely on the concepts that are usually associated with AB: designing the weight vector to minimizing the beamformer output while maintaining unity gain in the look direction. It is also different from the AB derivation based on subtracting a rank-1 matrix from the CSM while keeping the difference nonnegative. Instead, the new derivation seeks the source time history to conform with the steering vectors(s) and the array data using the criteria described.

Two passes through the methodology are given, one for scalar beamforming and one for coherent matrix beamforming. The new method is described as an AB derivation because it gives the AB formulas in the scalar case and plausible generalizations in the matrix case. It is possible that many beamforming problems can be addressed with this formulation.

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