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A MATHEMATICAL PERSPECTIVE ON ACOUSTIC ARRAY IMAGING METHODS

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Abstract

Microphone arrays are a widely used measurement device for the localization and quantification of (aero-)acoustic sources. Using the tools of mathematics, the common formulations of array imaging methods can be further formalized. In this conference paper we present two specific mathematical frameworks namely a representation of Beamforming functionals by characteristic scalar products and an infinite dimensional inverse problem on function spaces. For each aspect we illustrate how a theoretical analysis can be carried out and mathematical statements can be derived. Major focus is put on the analysis of the mean squared error of Beamforming functionals and the study of uniqueness of the inverse source problem.

1 INTRODUCTION

Interdisciplinary research projects of engineers and mathematicians are nowadays very common in many fields of research, one example being the simulation of physical phenomena such as fluid flow or solid mechanics. It may be somewhat less well-known, that many experimental methods provide very interesting and challenging mathematical problems as well. Often one aims to estimate an unknown quantity which cannot be measured directly. Instead, only observed data that are caused by the unknown quantity is accessible. Such scenarios, where we seek to infer information from an directly not observable state are usually referred to as *inverse problems*. The task of aeroacoustic source power reconstruction from correlation measurements belongs to this class of problems. With the tools of mathematics, this inverse problem can be analyzed on an abstract level. Thereby it is possible to prove mathematical statements on the properties of reconstruction methods or the problem itself. Two specific examples of such statements will be presented in this article:

1. Beamforming methods (see e.g. [7, 1]) are extensively used for the evaluation of aeroacoustic measurement data. In the last decades many variants of Beamforming techniques have been introduced. Among the most important are *Robust Adaptive Beamforming* [2], *Orthogonal Beamforming* [11], *Functional Beamforming* [3] and *Conventional Beamforming* (CBF) [7], whereby we limit our considerations in this paper to CBF. In Section 3, we study the CBF imaging functional, using an alternative representation based on scalar products on finite dimensional vector spaces. This structure embeds CFB into a whole class of Beamforming methods. Further we can quantify the mean squared error of the Beamforming solution.

2. In Section 4 we will present a mathematical formulation of the source reconstruction problem based on infinite dimensional spaces. With this structure it is possible to investigate whether acoustic source powers are uniquely determined by correlation measurements.

2 STANDARD PROBLEM FORMULATION

We recall the standard discrete formulation of the aeroacoustic source reconstruction problem (see e.g. [10]). The array consists of N microphones at postions $\mathbf{x}_i$ and the source domain is discretized by M focus points $\mathbf{y}_j$. From now on we will always indicate finite dimensional objects, that have an infinite dimensional counterpart by an underscore. The array signal $\mathbf{p} \in \mathbb{C}^N$ and source signal $\mathbf{s} \in \mathbb{C}^M$ are both considered as random signals. Source reconstruction methods in experimental aeroacoustics are usually correlation-based, i.e. the correlation matrix of the source and array signal serve as input and output data of the mathematical model. More precisely, measured data and source data are related by the following equation

$$\mathbb{E}\left\{\mathbf{p}\mathbf{p}^H\right\} = \underline{\mathbf{G}}\,\mathbb{E}\left\{\mathbf{s}\mathbf{s}^H\right\}\,\underline{\mathbf{G}}^H\ . \tag{1}$$

The sound propagation is modeled by the matrix $\underline{\mathbf{G}} \in \mathbb{C}^{N\times M}$ with an appropriate Green's function g

$$\underline{\mathbf{G}}_{ij} = g(\mathbf{x}_i, \mathbf{y}_j)\ . \tag{2}$$

Defining the exact cross spectral matrix (CSM) $\underline{\mathbf{C}} = \mathbb{E}\left\{\mathbf{p}\mathbf{p}^H\right\}$ and the source correlation matrix $\mathbf{S} = \mathbb{E}\left\{\mathbf{s}\mathbf{s}^H\right\}$, Equation (1) reads as

$$\underline{\mathbf{C}} = \underline{\mathbf{G}}\ \underline{\mathbf{S}}\ \underline{\mathbf{G}}^H\ . \tag{3}$$

Often, it is assumed that the sources are spatially uncorrelated, then (3) further reduces to

$$\underline{\mathbf{C}} = \underline{\mathbf{G}}\ \underline{\underline{\mathbf{M}_{\mathbf{q}}}}\ \underline{\mathbf{G}}^H\ , \tag{4}$$

with the vector of source powers

$$\mathbf{q} = \left(\mathbb{E}\left\{|\mathbf{s}_1|^2\right\},\quad \dots,\quad \mathbb{E}\left\{|\mathbf{s}_M|^2\right\}\right)^T \tag{5}$$

and the diagonal source power matrix $\underline{\mathbf{M}_{\mathbf{q}}} = \operatorname{diag}(\mathbf{q}_1, \ldots, \mathbf{q}_M)$.

3 ERROR ANALYSIS OF BEAMFORMING METHODS

In a real experimental setup, only a random approximation $\underline{\mathbf{C}}^{\text{obs}} \approx \mathbb{E}\left\{\mathbf{p}\mathbf{p}^H\right\}$ is available. A fast and robust estimator on the source power is provided by Conventional Beamforming (CBF) [7]. The CBF solution can be represented as the solution of a minimization problem (see [12])

$$J^{\text{cb}}(\mathbf{y}) = \operatorname*{argmin}_{A\in\mathbb{C}} \sum_{m=1}^{N}\sum_{n=1}^{N} \left|\underline{\mathbf{C}}^{\text{obs}}_{mn} - A\, \mathbf{g}(\mathbf{y})_m\, \overline{\mathbf{g}(\mathbf{y})_n}\right|^2 , \tag{6}$$

with the steering vector $\mathbf{g} \in \mathbb{C}^N$ defined by

$$\mathbf{g}(\mathbf{y}) = \begin{pmatrix} g(\mathbf{x}_1,\mathbf{y}), & \ldots, & g(\mathbf{x}_N,\mathbf{y}) \end{pmatrix}^T . \tag{7}$$

Defining the steering matrix $\mathbf{P}(\mathbf{y}) = \mathbf{g}(\mathbf{y})\mathbf{g}(\mathbf{y})^H$, the minimization problem (6) is equivalent to

$$J^{\text{cb}}(\mathbf{y}) = \operatorname*{argmin}_{A\in\mathbb{C}} \left\langle \underline{\mathbf{C}}^{\text{obs}} - A\mathbf{P}(\mathbf{y}), \underline{\mathbf{C}}^{\text{obs}} - A\mathbf{P}(\mathbf{y}) \right\rangle_F . \tag{8}$$

Here $\langle\cdot,\cdot\rangle_F$ denotes the Frobenius scalar product

$$\langle \mathbf{A}, \mathbf{B}\rangle_F = \sum_{m=1}^{N}\sum_{n=1}^{N} \mathbf{A}_{mn}\overline{\mathbf{B}_{mn}} \quad \text{for } \mathbf{A},\mathbf{B} \in \mathbb{C}^{N\times N} . \tag{9}$$

From now on the dependency on the focus point $\mathbf{y}$ will be omitted to increase readability. We recall the basic notion of a scalar product.

Definition 3.1 (Scalar product (see i.e. [9]))**.**
A mapping $\langle\cdot,\cdot\rangle : V\times V \to \mathbb{C}$ on a complex vector space V is called a scalar product if the following conditions are satisfied for all $x,y,z \in V$ and $\lambda \in \mathbb{C}$

1. $\langle x,x\rangle \geq 0$ *(positive)*
2. $\langle x,x\rangle = 0$ *iff* $x = 0$ *(definite)*
3. $\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$ *(additive)*
4. $\langle \lambda x,y\rangle = \lambda\langle x,y\rangle$ *(homogeneous)*
5. $\langle x,y\rangle = \overline{\langle y,x\rangle}$ *(conjugate symmetric)*

The following two fundamental properties of scalar products are required for the further analysis.

Remark 3.2. *(Properties of a scalar product (see i.e. [9]))*

- *A scalar product $\langle\cdot,\cdot\rangle$ defines a norm by* $\|x\| = \sqrt{\langle x,x\rangle}$.

- *For $x, y \in V$ the Cauchy-Schwarz inequality holds true*

$$|\langle x,y\rangle| \leq \|x\| \, \|y\| \ . \tag{10}$$

The Frobenius scalar product (9) can be generalized by introducing weighting coefficients. More precisely, every symmetric weighting matrix $\mathbf{W} \in \mathbb{R}^{N\times N}$ with purely positive entries defines a scalar product $\langle \cdot,\cdot\rangle_{\mathbf{W}}$ on $\mathbb{C}^{N\times N}$ by

$$\langle \mathbf{A},\mathbf{B}\rangle_{\mathbf{W}} = \langle \mathbf{W}\odot\mathbf{A},\mathbf{B}\rangle_F = \langle \mathbf{A},\mathbf{W}\odot\mathbf{B}\rangle_F \quad \text{for } \mathbf{A},\mathbf{B}\in\mathbb{C}^{N\times N} \ , \tag{11}$$

where $\odot$ denotes the elementwise or Hadamard-product. The properties of a scalar product are easily checked. For the choice $\mathbf{W} = \mathbf{1}$, where $\mathbf{1}$ denotes the matrix with all entries equal to 1, we obtain the Frobenius scalar product. The minimization problem (6) can now be generalized for the weighted scalar product $\langle \cdot,\cdot\rangle_{\mathbf{W}}$ by

$$J_{\mathbf{W}} = \underset{A\in\mathbb{C}}{\operatorname{argmin}} \left\langle \underline{\mathbf{C}}^{\text{obs}} - A\mathbf{P}, \underline{\mathbf{C}}^{\text{obs}} - A\mathbf{P} \right\rangle_{\mathbf{W}} \ . \tag{12}$$

Straightforward calculations show, that the solution to (12) is given by

$$J_{\mathbf{W}} = \frac{\left\langle \underline{\mathbf{C}}^{\text{obs}},\mathbf{P}\right\rangle_{\mathbf{W}}}{\langle \mathbf{P},\mathbf{P}\rangle_{\mathbf{W}}} \ . \tag{13}$$

Now we consider a specific choice for the weighting matrix $\mathbf{W}$, given by the inverse variances of the corresponding entry of the random CSM estimator $\underline{\mathbf{C}}^{\text{obs}}$

$$\mathbf{W}^{\text{iv}}_{ml} = \sigma_{ml}^{-2} = \operatorname{Var}\left(\underline{\mathbf{C}}^{\text{obs}}\right)^{-1} \ . \tag{14}$$

The CSM-variances σ_{ml}^2 may be estimated by sample variances (for more details on the inverse variance Beamforming method, we refer to [8]). For the following analysis we further assume that the entries of the CSM estimator are mutually uncorrelated i.e. that

$$\operatorname{Cov}\left(\underline{\mathbf{C}}^{\text{obs}}_{mn}, \underline{\mathbf{C}}^{\text{obs}}_{m'n'}\right) = 0 \quad \text{for } (m,n) \neq (m',n') \ . \tag{15}$$

Since $\underline{\mathbf{C}}^{\text{obs}}$ is a random quantity, the Beamforming estimators $J_{\mathbf{W}}$ are random as well. The mean squared error (MSE), defined by

$$\text{MSE}_{\mathbf{W}} = \operatorname{Var}(J_{\mathbf{W}}) \ , \tag{16}$$

is an important statistical error measure for such estimators. For inverse variance Beamforming we derive the following MSE

$$\text{MSE}_{\mathbf{W}^{\text{iv}}} = \operatorname{Var}(J_{\mathbf{W}^{\text{iv}}}) \overset{(i)}{=} \frac{\operatorname{Var}\left(\sum\limits_{m,n} \frac{1}{\sigma_{mn}^2} \underline{\mathbf{C}}^{\text{obs}}_{mn} \overline{\mathbf{P}_{mn}}\right)}{\langle \mathbf{P},\mathbf{P}\rangle^2_{\mathbf{W}^{\text{iv}}}} \overset{(ii)}{=} \frac{\sum\limits_{m,n} \frac{1}{\sigma_{mn}^2} |\mathbf{P}_{mn}|^2}{\langle \mathbf{P},\mathbf{P}\rangle^2_{\mathbf{W}^{\text{iv}}}} \overset{(iii)}{=} \frac{1}{\langle \mathbf{P},\mathbf{P}\rangle_{\mathbf{W}^{\text{iv}}}} \ . \tag{17}$$

Comments on the intermediate steps of (17):

(i) Definition of $J_{\mathbf{W}^{\mathrm{iv}}}$ and the property of the variance $\mathrm{Var}(aX) = |a|^2 \mathrm{Var}(X)$ for any $a \in \mathbb{C}$.

(ii) Assumption (15) implies, that the variance of the sum equals the sum of the variances.

(iii) The denominator equals the square of the nominator and thus, the nominator cancels out.

For the MSE of CBF we obtain the following estimate

$$\begin{aligned} \mathrm{MSE}_{\mathbf{1}} = \mathrm{Var}(J_{\mathbf{1}}) = \frac{\mathrm{Var}\left(\sum\limits_{m,n} \underline{\mathbf{C}}^{\mathrm{obs}}_{mn} \overline{\mathbf{P}_{mn}}\right)}{\langle \mathbf{P}, \mathbf{P} \rangle_{\mathbf{1}}^2} &\overset{(i)}{=} \frac{\sum\limits_{m,n} \sigma_{mn}^2 |\mathbf{P}_{mn}|^2}{\left(\sum\limits_{m,n} \frac{\sigma_{mn}}{\sigma_{mn}} |\mathbf{P}_{mn}|^2\right)^2} \\ &\overset{(ii)}{\geq} \frac{\sum\limits_{m,n} \sigma_{mn}^2 |\mathbf{P}_{mn}|^2}{\left(\sum\limits_{m,n} \sigma_{mn}^2 |\mathbf{P}_{mn}|^2\right)\left(\sum\limits_{m,n} \frac{1}{\sigma_{mn}^2} |\mathbf{P}_{mn}|^2\right)} \overset{(iii)}{=} \frac{1}{\langle \mathbf{P}, \mathbf{P} \rangle_{\mathbf{W}^{\mathrm{iv}}}} . \end{aligned} \tag{18}$$

Comments on the intermediate steps of (18):

(i) For the nominator we used Assumption (15) again. For the denominator we multiplied each summand with a unity factor.

(ii) We replaced the denominater $\big($scalar product of $\sigma_{mn} |\mathbf{P}_{mn}|$ and $\sigma_{mn}^{-1} |\mathbf{P}_{mn}|\big)^2$ by a quantity that is greater or equal $\big($squared norm of $\sigma_{mn} |\mathbf{P}_{mn}|$ times squared norm of $\sigma_{mn} |\mathbf{P}_{mn}|\big)$. Where the indexed quantities represent the matricies with the corresponding entries. The inequality is valid due to the Cauchy-Schwarz inequality (10).

(iii) The expression simplifies since the nominator equals the first factor in the denominator. The remaining term in the denominator equals the squared $\mathbf{W}^{\mathrm{iv}}-$norm.

This shows that the MSE of the CBF estimator can be bounded from below by the MSE of the inverse variance estimator which proves the following mathematical statement.

Lemma 3.3 (MSE relation of CBF and inverse variance Beamforming)**.**
Assume that the entries of the observed CSM are mutually uncorrelated (see (15)*). Then the error inequality*

$$\mathrm{MSE}_{\mathbf{W}^{iv}} \leq \mathrm{MSE}_{\mathbf{1}}$$

holds true.

4 THE INVERSE SOURCE PROBLEM ON FUNCTION SPACES

This section illustrates some parts of the results that have been presented in greater detail in [6]. As already mentioned in the introduction, the sound source reconstruction problem belongs to the class of inverse problems. If the unknown quantity of interest is a source term, the notion *inverse source problem* is often used. A generic inverse problem can be described by an operator equation

$$\text{find } f \text{ s.t.} \quad F(f) = h, \tag{19}$$

where F denotes the forward operator modelling the physics of the problem and h the observed data. For many real world problems, the solution space is infinite dimensional (e.g. a function space) whereas the data space is finite dimensional. The inverse source problem we consider in aeroacoustics has exactely these properties: the desired solution is a source power function, defined on the source region and the (exact) data $\underline{\mathbf{C}} \in \mathbb{C}^{N\times N}$ is finite. Given a specific inverse problem one may ask if a solution is uniquely determined or if there may exist $f_1 \neq f_2$ such that

$$F(f_1) = F(f_2)\ , \tag{20}$$

i.e. two different states that generate the same data. For the aeroacoustic inverse source problem it is intuitive, that the reconstruction of a source power function q by data from finitely many microphones is ambiguous i.e. that two different source power functions can generate the same cross spectral matrix. However, it is not clear if this ambiguity occurs only because of the discrete data space or if it is an intrinsic property of the inverse source problem. This issue can be clarified by an investigation of the infinite dimensional problem formulation. It is obtained by a straightforward generalization of Equation (4) to function spaces. Defining the measurement region $\mathbb{M}$ and the source region Ω we get

$$\mathbf{C} = \mathbf{G}\mathbf{M}_q\mathbf{G}^* =: \mathscr{C}(q)\ . \tag{21}$$

The source power q is now a function in the space of square integrable functions on the source region (denoted by $L^2(\Omega)$). The measured data are represented by a covariance operator $\mathbf{C}$ on the space of square integrable functions on the measurement region (denoted by $L^2(\mathbb{M})$). Note that the forward operator $\mathscr{C}$ maps a function $q \in L^2(\Omega)$ to an operator i.e. $\mathscr{C}(q)$ is an operator itself. The sound propagation operator $\mathbf{G}$ maps functions from $L^2(\Omega)$ to functions on $L^2(\mathbb{M})$ and the source multiplication operator $\mathbf{M}_q$ denotes the multiplication by the source power function q. More precisely we define:
$\mathbf{G} : L^2(\Omega) \to L^2(\mathbb{M})$ (sound propagation operator)

$$(\mathbf{G}v)(\mathbf{x}) = \int_\Omega g(\mathbf{x},\mathbf{y})v(\mathbf{y})d\mathbf{y}\ .$$

$\mathbf{G}^* : L^2(\mathbb{M}) \to L^2(\Omega)$ (adjoint of the sound propagation operator)

$$(\mathbf{G}^*w)(\mathbf{y}) = \int_\mathbb{M} \overline{g(\mathbf{x},\mathbf{y})}w(\mathbf{x})d\mathbf{x}\ .$$

M_q (source multiplication operator)

$$(M_q v)(\mathbf{y}) = q(\mathbf{y})v(\mathbf{y})\ .$$

We want to emphasize that the inverse source problem we study, considers stochastic and uncorrelated sources. In contrast to (21), one may also study deterministic (and therefore completely correlated) sources, which leads to the problem formulation

$$\text{given } w \text{ find } v \text{ s.t.} \quad \mathbf{G}v = w \,. \tag{22}$$

It is well known (see e.g. [4, 5]) that problem (22) is not uniquely solvable since there exist so called non-radiating sources. For v beeing a non-radiating source, $\mathbf{G}v$ vanishes everywhere outside of the source region Ω.

Using the representation given by Equation (21), we will sketch how the uniqueness of source power functions can be studied. We will concentrate on the main ideas without diving too deep into advanced mathematics. For a full proof for free field sound propagation in $\mathbb{R}^2$ and $\mathbb{R}^3$ within a homogeneous subsonic flow field we refer to [6]. For now we consider Green's function in $\mathbb{R}^3$ without flow given by

$$g(\mathbf{x},\mathbf{y}) = \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \,. \tag{23}$$

For any $\hat{\mathbf{x}} \in \mathbb{R}^3$ with unit length we define

$$u_{\hat{\mathbf{x}}}(\mathbf{y}) = e^{ik\hat{\mathbf{x}}\cdot\mathbf{y}} \,. \tag{24}$$

Those planes wave functions are essential for proving the uniqueness result of the following theorem. It states that bounded source power functions are uniquely determined by their correlation data, where the space of bounded functions on the source region is denoted by $L^\infty(\Omega)$.

Theorem 4.1 (Uniqueness)**.**
Assume that $q_1, q_2 \in L^\infty(\Omega)$ and

$$\mathscr{C}(q_1) = \mathscr{C}(q_2)$$

then q_1 and q_2 coincide, i.e. $q_1 = q_2$.

Proof. See [6, Theorem 3.6]. □

For the remainder of this section, we present the key ideas for proving Theorem 4.1:
Since $\mathscr{C}$ is linear, we can equivalently show that $\mathscr{C}(q) = 0$ implies $q = 0$ or in other words: whenever $\mathscr{C}(q)$ is the zero operator, the function q must be the zero function.

The assumption of $\mathscr{C}(q)$ being the zero operator eventually (after some technical substeps) yields

$$0 = \int_\Omega q(\mathbf{y})u_1(\mathbf{y})\overline{u_2(\mathbf{y})}d\mathbf{y} \quad \text{for all } u_1, u_2 \in \mathscr{S} \,. \tag{25}$$

Where $\mathscr{S}$ denotes a set of functions that won't be specified here. It turns out that for two unit length vectors $\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2$ the plane waves $u_{\hat{\mathbf{x}}_1}, u_{\hat{\mathbf{x}}_2}$ (see (24)) belong to $\mathscr{S}$. Therefore we can insert them in Equation (25), which yields

$$0 = \int_\Omega q(\mathbf{y})e^{-ik(\hat{\mathbf{x}}_2-\hat{\mathbf{x}}_1)\cdot\mathbf{y}}d\mathbf{y} = (\mathscr{F}q)(\hat{\mathbf{x}}_2 - \hat{\mathbf{x}}_1) \,, \tag{26}$$

where $\mathscr{F}$ denotes the Fourier transform. The set of vectors $\mathbf{z} \in \mathbb{R}^3$ that can be represented by the difference of two vectors with unit length is exactly the ball with radius 2

$$B = \left\{ \mathbf{z} \in \mathbb{R}^3 \middle| \ |\mathbf{z}| \leq 2 \right\} . \tag{27}$$

So by (26) we have the property

$$(\mathscr{F}q)(\mathbf{z}) = 0 \quad \text{for all } \mathbf{z} \in B. \tag{28}$$

Using some results from the theory of complex functions we can show that if the Fourier transform of q vanishes on the ball B it must vanish everywhere i.e.

$$(\mathscr{F}q)(\mathbf{z}) = 0 \quad \text{for all } \mathbf{z} \in \mathbb{R}^3. \tag{29}$$

Since the Fourier transform of q is the zero function, q must be itself the zero function. This completes the proof.

5 CONCLUSION

In this conference contribution we have illustrated research collaborations between engineers and mathematicians. In particular, we focused on mathematical modelling and the derivation of mathematical statements. We would like to point out that such interdisciplinary collaborations can be beneficial for both fields of research. The engineering community provides a hard inverse problem of actual interest and many interesting results on source reconstruction methods. On the other hand, the tools of advanced mathematics can provide results that are complementary to those from an experimental point of view. If the findings are combined, they may help to deepen the understanding of reconstruction results from experimental data and further develop array imaging methods for (aero-)acoustic purposes. Finally we discuss the two abstract results (Lemma 3.3 and Theorem 4.1) from the perspective of applied science.

1. Section 3 points out how the reliability of Beamforming results regarding repetitions of the experiment can be quantified. Methods with a lower MSE are expected to show less variations in their output, if the experiment is repeated several times. Inverse variance Beamforming is a modified Beamformer, that can be implemented with the same computational complexity as CBF and also reduces the MSE.

2. Loosely speaking, the uniqueness result from Section 4 states that it is not hopeless to recover the source power function from correlation measurements. Although it may not be possible to measure with infinitely many microphones, using more microphones should improve the reconstruction result if the theoretical assumptions are valid. However, this statement is not as trivial as it may appear at first glance. If the infinite dimensional problem wasn't uniquely solvable, the fact of measuring correlations outside of the source region would imply an irretrievable loss of source power data.

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