WHAT IS BEAMFORMING ?

Robert P. Dougherty

OptiNav, Inc.
10900 NE 8TH ST, Suite 900, Bellevue, WA, USA

ABSTRACT

Beamforming is an imaging technique that has found many applications in aeroacoustics, and continues to evolve to meet greater challenges. It has elements in common with other methods such as nearfield acoustic holography, but its strength is distributed, broadband, incoherent sources at arbitrary distance from the array. The formulation of the classical technique in the frequency domain is simple and lends itself to many types of analysis. A derivation is given here that leads to an expression for the variance of the beamform map when the integration time is finite and not all of the elements of the cross spectral matrix are included.

NOMENCLATURE

$A_j(\vec{\xi})$ = Point Spread Function (PSF) for a source at $\vec{\xi}_j$ when beamforming to $\vec{\xi}$
b = beamform map
C = Cross Spectral Matrix (CSM)
$\overline{\mathbf{C}}$ = trimmed CSM, e.g., after diagonal deletion
E(x) = Expectation value of x
$e^{-i\omega t}$ = Fast phase factor divided out of all complex pressure quantities
g = steering vector
g_n = an element of a steering vector
j, k = beamform grid or source point indices
m, n = microphone indices
M = number of acoustic sources
N = number of array microphones
N_I = number of blocks $= T\Delta\nu$
N_S = number of elements of S
p_n = complex narrowband unsteady pressure measured at microphone n
$q(\vec{\xi}_j)$ = complex narrowband time history of a source at $\vec{\xi}_j$
R_n = complex self noise at microphone n
r_n = microphone self noise power $= E(|R_n|^2)$
S = the set of microphone pairs included in the cross spectral matrix for beamforming
s_j = power of source j = $E(q_j^* q_j)$
T = integration time
u = steered array data
w = weight vector
$\vec{x}_n$ = location of microphone n
Δt = block length $= \frac{1}{\Delta\nu} = \frac{T}{N_I}$
α = weight vector normalization coefficient
$\Delta\nu$ = analysis bandwidth
$\vec{\xi}$ = point in the beamform grid
$\vec{\xi}_j$ = location of source j
$\langle f \rangle$ = time average $\langle f \rangle = \frac{1}{N_I}\sum_{l=1}^{N_I} f(\text{time block } l)$
$\mathbf{z}'$ = Hermitian conjugate (complex conjugate transpose) of $\mathbf{z}$

1 INTRODUCTION

Aeroacoustic beamforming is a method for processing microphone array data to produce images that represent the distribution of the acoustic source strength. It is an imaging technique that applies to continuous or discrete source distributions. The distance from the source region to the array is not restricted. The resolution is governed by the same Rayleigh formulas that govern diffraction-limited optics. Superresolution algorithms that can potentially locate sources to the theoretical limit of the Cramer-Rao bound have been defined, but are restricted in applicability. The aeroacoustic application requires the array to operate over a very wide frequency range compared with electromagnetic beamforming. Grating lobes are prevented by applying sparse, wideband microphone arrangements. These come with a drawback of a number of sidelobes in addition the sidelobes related to the overall aperture shape. Efforts to determine component spectra for subregions of the beamforming grid or to image sources far below the highest source in level must be able to compensate for the sidelobes. Classical beamforming gives the best results for incoherent, broadband, source distributions. Airframe noise measurement provides an excellent match to the strengths of the technique. Another strong selling point is the ability of beamforming to locate rogue sources; sources that are not expected and would potentially contaminate the results of conventional microphone measurements. Numerous extensions to the technique, in addition to superesolution beamforming have been developed and are continuing to appear. Methods have been developed for dealing with uniform and nonuniform flow effects, reverberant environments, linearly and nonlinearly moving sources, pressurized wind tunnels, for fusing optical and acoustic images, and for maintaining constant resolution over a range of frequencies. Deconvolution techniques attempt to extract the true source distribution by removing some of the artifacts introduced by the array. Notable examples include the DAMAS and CLEAN-SC deconvolution techniques. Current research includes finding ways to accurately represent extended, coherent, source distributions, beamform in complex, small environments such as turbofan engine nacelles and automobile cabins, and expand the beamforming space to include independent parameters in addition to frequency and spatial coordinates, as well as multipole source distributions. New array designs include multiple sensor modes. A guide star method has been developed to remove effects of turbulent decorrelation, but this remains a big challenge in beamforming. Instrumentation and data management are also continuing issues in beamforming. Since the results improve with the number of channels, the budget for microphones and data acquisition systems is often a limiting factor. A number of references can be found in the short review article [1] and the book [2].

This paper presents the classical beamforming algorithm in the frequency domain using an extension of the compact notation in [3] while filling in a few of the details. The derivation is different from the one given in [3], emphasizing an intuitive imaging process rather than an optimization problem. Reference [3] continues from classical beamforming to discuss several important deconvolution algorithms. The only ambitious goal in this paper is to derive a formula for the variance of the beamforming result for the case of finite integration time and a partial ("trimmed" in the terminology of [3]) cross spectral matrix.

2 PROBLEM FORMULATION

2.1 Source-receiver model

Consider an array of N microphones and a beamform grid (Fig. 1.) The Green's function for grid point $\vec{\xi}$ and microphone n is $g_n(\vec{\xi})$. An example is $g_n(\vec{\xi}) = e^{ik|\vec{\xi}-\vec{x}_n|} / |\vec{\xi} - \vec{x}_n|$.

The model for the pressure is $\mathbf{p}$ is

$$\mathbf{p} = \sum_{j=1}^{M} q_j \mathbf{g}(\vec{\xi}_j) + \mathbf{R}, \tag{1}$$

where q_j is the time history of source j, and $\mathbf{R}$ is the microphone self (flow) noise. The pressure is recorded for a time T and divided into N_I (conceptually non-overlapping) blocks of length Δt. An FFT is applied to each block, giving an analysis bandwidth of $\Delta\nu = \frac{1}{\Delta t}$. There are N_I data vectors, $\mathbf{p}$. Each source j has power s_j and N_I source time history values, q_j, that enter into the model. These are assumed to be zero-mean, random, and mutually incoherent:

$$\mathrm{cov}(q_j^*, q_k) = s_j \delta_{jk}. \tag{2}$$

The self noise components, R_n, have power $r_n = \mathrm{var}(|R_n|)$, $n = 1,\ldots,$ N, and are assumed to mutually incoherent and uncorrelated with the acoustic sources.

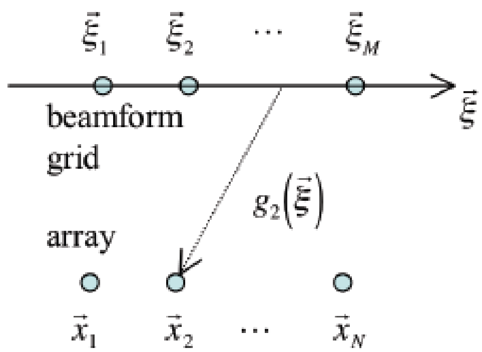

Fig. 1. A beamform grid and a phased array of microphones

2.2 Beam steering

The array is steered to $\vec{\xi}$ by forming the N_I complex numbers (α is determined below)

$$u(\vec{\xi}) = \alpha \mathbf{g}'(\vec{\xi}) \mathbf{p} = \alpha \sum_{n=1}^{N} g_n'^*(\vec{\xi}) p_n, \tag{3}$$

The function $u(\vec{\xi})$ is intended to be similar to the source time history for the point $\vec{\xi}$:

$$u(\vec{\xi}) = \alpha\left[\sum_{j=1}^{M} q_j \mathbf{g}'(\vec{\xi})\mathbf{g}(\vec{\xi}_j) + \mathbf{g}'(\vec{\xi})\mathbf{R}\right]. \quad (4)$$

The array is designed so that $\mathbf{g}(\vec{\xi})$ varies strongly with $\vec{\xi}$. Ideally $\mathbf{g}'(\vec{\xi})\mathbf{g}(\vec{\xi}_j)$ will have peak at $\vec{\xi} = \vec{\xi}_j$ since the inner product of a vector with itself is gives a maximum. For source *k*:

$$u(\vec{\xi}_k) = \alpha\left[q_k\left\|\mathbf{g}(\vec{\xi}_k)\right\|^2 + \sum_{j \neq k} q_j \mathbf{g}'(\vec{\xi}_k)\mathbf{g}(\vec{\xi}_j) + \mathbf{g}'(\vec{\xi}_k)\mathbf{R}\right]. \quad (5)$$

2.3 Source strength maps images

The average power of Eq. 3 is

$$b(\vec{\xi}) = \left\langle\left|u(\vec{\xi})\right|^2\right\rangle = \alpha^2\left\langle\left|\mathbf{g}'(\vec{\xi})\mathbf{p}\right|^2\right\rangle = \alpha^2\mathbf{g}'(\vec{\xi})\langle\mathbf{pp}'\rangle\mathbf{g}(\vec{\xi}) = \alpha^2\mathbf{g}'(\vec{\xi})\mathbf{C}\mathbf{g}(\vec{\xi}), \quad (6)$$

where the last form introduces the array Cross Spectral Matrix (CSM)

$$\mathbf{C} = \langle\mathbf{pp}'\rangle. \quad (7)$$

Substituting Eq. 1 into Eq. 6 for a single source and no self noise gives

$$s_j = b(\vec{\xi}_j) = \alpha^2 s_j \mathbf{g}'(\vec{\xi})\mathbf{g}(\vec{\xi}_j)\mathbf{g}'(\vec{\xi}_j)\mathbf{g}(\vec{\xi}_j). \quad (8)$$

Solving for α gives

$$\alpha = \frac{1}{\sqrt{(\mathbf{g}'\mathbf{g})^2}} = \frac{1}{\sqrt{\sum_{m,n}|g_m|^2|g_n|^2}}, \quad (9)$$

Defining the array weight vector by $\mathbf{w}(\vec{\xi}) = \alpha\mathbf{g}(\vec{\xi})$, Eq. 6, can be rewritten

$$b(\vec{\xi}) = \mathbf{w}'(\vec{\xi})\mathbf{C}\mathbf{w}(\vec{\xi}) \quad \text{(Classical beamforming expression)} \quad (10)$$

It is often the case that some of the elements of the CSM do more harm than good in beamforming. As show below, for example, the diagonal elements simply add a noise floor to the beamform map [2]. Also, certain elements are deleted when using a cross-shaped array [4]. The “trimmed” CSM, $\overline{\mathbf{C}}$ [3] has elements $S = \{(m,n)|C_{mn} \text{ is not set to } 0\}$. This gives

$$b(\vec{\xi}) = \mathbf{w}'(\vec{\xi})\overline{\mathbf{C}}\mathbf{w}(\vec{\xi}) \qquad \alpha = \frac{1}{\sqrt{\sum_{(m,n)\in S}|g_m|^2|g_n|^2}}. \quad (11)$$

3 ANALYSIS

3.1 Expectation value of the beamform map

Using the statistical assumptions, the expectation value of the beamform map becomes

$$E\left[b(\vec{\xi})\right] = \sum_{j=1}^{M} A_j(\vec{\xi})s_j + \sum_{(m,n)\in S} w_m^*(\vec{\xi})w_n(\vec{\xi})\delta_{mn} r_n, \quad (11)$$

where the array point spread function for a source at $\vec{\xi}_j$ is given by

$$A_j\left(\vec{\xi}\right)=\sum_{(m,n)\in S} w_m^*\left(\vec{\xi}\right) g_m\left(\vec{\xi}_j\right) g_n^*\left(\vec{\xi}_j\right) w_n\left(\vec{\xi}\right) \quad A_j\left(\vec{\xi}_j\right)=1. \tag{12}$$

3.2 Variance

Deleting the diagonal elements of the CSM completely removes the microphone self noise from the expectation value of the beamform map. To choose the integration time, suppose that the cross-terms between the acoustic source and the self noise can be neglected relative to the cross terms between self noise at different microphones. Then

$$b\left(\vec{\xi}\right)=\left\langle\left|q_j\right|^2\right\rangle A_j\left(\vec{\xi}_j\right)+\sum_{(m,n)\in S} w_m^*\left(\vec{\xi}\right) w_n\left(\vec{\xi}\right)\left\langle R_m R_n^*\right\rangle. \tag{13}$$

Assuming the acoustic source is Gaussian broadband noise and that the data blocks are independent, manipulation of Eq. 13 gives

$$\operatorname{var}\left(\left\langle\left|q_j\right|^2\right\rangle\right)=\frac{2\left[\operatorname{var}\left(\left|q_j\right|\right)\right]^2}{N_I}=\frac{2s_j^2}{N_I} \quad \text{and} \tag{14}$$

$$\operatorname{var}\left[\sum_{(m,n)\in S} w_m^*\left(\vec{\xi}\right) w_n\left(\vec{\xi}\right)\left\langle R_m R_n^*\right\rangle\right]=\frac{N_S}{N_I} w^2 r^2 \tag{15}$$

where N_S is the number of elements of S and w and r are the magnitudes of weight vector elements and the self noise power, respectively (assumed uniform over the array).

In terms of the pressure at the array, the variance of the beamforming peak simplifies to

$$\operatorname{var}\left(\left|p_{bp}\right|^2\right)=\frac{2p^2}{N_I}+\frac{r^2}{N_I}. \tag{16}$$

4 SUMMARY

Beamforming is a powerful, flexible, and continuously evolving measurement technique in aeroacoustics. A derivation of the classical formulation has been given, including formulas giving the variance of the result in the practical case of finite integration time.

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